HOMEWORK 2 EXTRA EXERCISE (AND THINGS TO THINK ABOUT)

Question. Why is the partial fraction decomposition of a proper rational function what it is?

That was a great question I got after class from a student. Let's try to think about this question a little bit. We saw in examples (in class and in the homework) that the partial fraction decomposition that I claimed exists does indeed work. Let's also maybe see in examples why something else wouldn't work and then try to think about how one might have come up with the partial fraction decomposition rule.

Exercise 1. (a) Show that the following decomposition cannot hold

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+4)^2}$$

for constants A, B, C.

(b) Show that the following decomposition cannot hold

$$\frac{5x-7}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{x-1} + \frac{C}{x-1}$$

for constants A, B, C.

Hopefully you have grown more and more convinced that you do need in your partial fraction decomposition all the increasing powers of each irreducible factor in the denominator up to the highest power with which it appears in the factorization of the denominator.

Here is a sketch of why that is that I invite you to read if you want to understand the process more:

- If $Q(x) = q_1(x)q_2(x)$ where $q_1(x)$ and $q_2(x)$ are relatively prime factors (i.e. they have no common factors up to constant), then there exist polynomials $a(x)$ and $b(x)$ such that
$$\frac{P(x)}{Q(X)} = \frac{P(x)}{q_1(x)q_2(x)} = \frac{P(x)a(x)}{q_1(x)} + \frac{P(x)b(x)}{q_(x)}.$$
- If the denominator $Q(x) = c(x)^m$, a power of a polynomial $c(x)$ that does not apply directly. But we can divide $P(X)$ as in the division algorithm that we discussed in class to get
$$P(X) = q_0(x)c(x) + r_0(x)$$
where $q_0(x)$ si the quotient and $r_0(x)$ is the remainder. Now we can repeat this and divide $q_0(x)$ again by $c(x)$ to get
$$q_0(x) = q_1(x)c(x) + r_1(x).$$

Combined, these give

$$P(x) = q_1(x)c(x)^2 + r_1(x)c(x) + r_0(x).$$

Repeat this process and you'll end up with

$$P(x) = q_{m-1}(x)c(x)^m + r_{m-1}(x)c(x)^{m-1} + \cdots + q_1(x)c(x)^2 + r_1(x)c(x) + r_0(x)$$

where each $r_i(x)$ has degree less than $c(x)$ since it is a remainder upon division by $c(x)$. Now dividing by $Q(x) = c(x)^m$, we get the partial fraction decomposition

$$P(x)/Q(x) = q_{m-1}(x) + \frac{r_{m-1}(x)}{c(x)} + \cdots + \frac{r_1(x)}{c(x)^{m-1}} + \frac{r_0(x)}{c(x)^m}.$$

So there you go; we saw in this example how all powers up to m show up in the denominators of the partial fraction decomposition

- Now you can decompose an arbitrary given denominator $Q(x)$ into a product of monic irreducible linear or quadratic terms to certain powers. By combining the previous two bullet points is how you arrive to the partial fraction decomposition rule we claimed in class and have used throughout the homework.